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Power Law Scaling of the Top Lyapunov Exponent of a Product of Random Matrices

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A sequence of i.i.d. matrix-valued random variables $\{X_n\} \cdot X_n = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ with probability p and $X_n = \begin{pmatrix} 1+a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & 1+a(\varepsilon) \end{pmatrix}$ with probability $1-p$ is considered. Let $a(\varepsilon) = a_0\varepsilon + o(\varepsilon)$, $c(\varepsilon) = c_0\varepsilon + o(\varepsilon)$ $\lim_{\varepsilon \to 0} b(\varepsilon) = 0$, a_0, c_0, $\varepsilon > 0$, and $b(\varepsilon) > 0$ for all $\varepsilon > 0$. It is shown show that the top Lyapunov exponent of the matrix product $X_n X_{n-1} \cdots X_1$, $\lambda = \lim_{n \to \infty}(1/n) | n \| X_n X_{n-1} \cdots X_i \|$ satisfies a power law with an exponent 1/2. That is, $\lim_{\varepsilon \to 0}(\ln \lambda / \ln \varepsilon) = 1/2$.

KEY WORDS: Lyapunov exponent; product of random matrices; Markov chain.

1. INTRODUCTION

Consider a sequence $\{X_n\}$ of matrix-valued, independent, identically distributed random variables, where

$$X_n = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = A \quad \text{with probability } p$$

and

$$X_n = \begin{pmatrix} 1+a(\varepsilon) & b(\varepsilon) \\ c(\varepsilon) & 1+a(\varepsilon) \end{pmatrix} = B \quad \text{with probability } 1-p$$

A and B are real, positive matrices and

$$\lim_{\varepsilon \to 0} a(\varepsilon) = \lim_{\varepsilon \to 0} c(\varepsilon) = 0, \qquad a(\varepsilon), c(\varepsilon) \geqslant 0 \cdot \lim_{\varepsilon \to 0} b(\varepsilon) = b_0 > 0$$

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A is a parabolic matrix, while B is a hyperbolic matrix, which is a perturbation of the parabolic matrix

$$\begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}$$

It is easy to see that B has two distinct eigenvalues μ_1, μ_2, while A has a multiple eigenvalue $\mu = 1$. Let

$$\lambda(\varepsilon) = \lim_{n \to \infty} \frac{1}{n} \ln \|X_n X_{n-1} \cdots X_1\|$$

be the top Lyapunov exponent of the random matrix product. Existence of λ is guaranteed by well-known theorems about products of random matrices.(1) It is easily seen that $\lambda(0) = 0$. In this paper I show that

$$\lim_{\varepsilon \to 0} \frac{\ln \lambda(\varepsilon)}{\ln \varepsilon} = \frac{1}{2}$$

Random matrix products where matrices in the product are perturbations of a parabolic matrix arise in the study of planar billiard problems.(2) A power law scaling of the top Lyapunov exponent with an exponent of 1/2 was proved for a large class of planar billiards in a recent paper by Wojtkowski.(3) Random matrix products of the type considered in the present paper (where the distribution of X_1 is supported on an uncountable set of parabolic and hyperbolic matrices) arose in the study of a billiard in a gravitational field. A power law scaling of the top Lyapunov exponent with an exponent of 1/2 was established numerically by Lehtihet and Miller.(4) Miller and Ravishankar(5) considered a stochastic model for the billiard in a gravitational field and showed that λ scales like ε^{α}, $1/2 \leqslant \alpha \leqslant 1$.

I prove the scaling of the Lyapunov exponent by establishing upper and lower bounds which scale like $\varepsilon^{1/2}$. A lower bound which scales like $\varepsilon^{1/2}$ can be obtained by using general results of Wojtkowski for products of random matrices.(5) I establish the lower bound using an elementary probabilistic argument, which I feel makes the result transparent for this particular problem. Results obtained in this paper can be easily extended to the case $d = d(\varepsilon)$, $\lim_{\varepsilon \to 0} d(\varepsilon) > 0$. One can further extend the result to the case when $d < 0$ and $b, c < 0$ by making the coordinate transformation $x' = x$, $y' = -y$.

2. SOME PROPERTIES OF *A* AND *B*

Assume that $a(\varepsilon) = a_0 \varepsilon + o(\varepsilon)$ and $c(\varepsilon) = c_0 \varepsilon + o(\varepsilon)$, $\varepsilon > 0$. [If one assumes $a \sim a_0 \varepsilon^{\alpha}$ and $c \sim c_0 \varepsilon^{\gamma}$, then the arguments given here will give a scaling exponent of $\min(\alpha, \gamma/2)$.] It is easy to see that $\mu_1 = (1+a) - (bc)^{1/2}$ and $\mu_2 = (1+a) + (bc)^{1/2}$ and the corresponding eigenvectors have slopes $-(c/b)^{1/2}$ and $(c/b)^{1/2}$, respectively. The action of B on a ray in $\mathbb{R}^2$ (a straight line through the origin) is to rotate it toward the expanding direction (eigendirection of μ_2) A rotates a ray in $\mathbb{R}^2$ in the clockwise direction The X axis is the eigendirection of A. From these observations it follows that the cone formed by the expanding direction and the X axis is left invariant by the actions of either matrix. Also note that as $\varepsilon \to 0$ the invariant cone collapes onto the X axis. Let us denote the slope of the expanding direction $(c/b)^{1/2}$ by m_E. Let σ be the invariant cone. Define a set of conical subsets of σ as follows:

$$\sigma_K = \left\{ V \in \mathbb{R}^2 \,\middle|\, \frac{m_E}{K} \leqslant \text{slope of } V \leqslant m_E \right\}$$

For a 2×2 matrix X define the norm $\|\cdot\|_{\sigma_K}$ as

$$\|X\|_{\sigma_K} = \operatorname{Sup}\{|XV|: V \in \sigma_K, |V| = 1\}$$

where $|V|$ is the Euclidean norm of V. It is easy to see that there exist constants $K_1, K_2 > 0$ such that

$$\|A\|_\sigma = \operatorname{Sup}\{|AV|: V \in \sigma, |V| = 1\}, |V| = 1\} \leqslant 1 + K_1 \varepsilon^{1/2}$$

$$\|B\|_\sigma \leqslant 1 + K_2 \varepsilon^{1/2}$$

With a little more effort one can also establish that for every $K \in N$, there exist positive constants $l_1(K)$ and $l_2(K)$ such that

$$\|A\|_{\sigma_K} \geqslant 1 + l_1 \varepsilon^{1/2}, \qquad \|B\|_{\sigma_K} \geqslant 1 + l_2 \varepsilon^{1/2} \tag{1}$$

(we assume $\varepsilon < 1$).

2.1. Upper Bound

We observe that for a.e. w (sequence of X_i) there exists an $n(w) \in N$ such that

$$(X_n X_{n-1} \cdots X_1) V \in \sigma \qquad \text{for all} \quad V \in \mathbb{R}^2 \tag{2}$$

From this it follows that

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \ln \|X_N X_{N-1} \cdots X_1\|$$

$$\leqslant \lim_{N \to \infty} \frac{1}{N} \ln \|X_N \cdots X_n\|$$

$$\leqslant p \ln \|A\|_\sigma \mu(1-p) \ln \|B\|_\sigma$$

for a.e. w. Therefore

$$\lambda \leqslant p \ln(1+K_1 \varepsilon^{1/2}) + (1-p) \ln(1+K_2 \varepsilon^{1/2})$$
$$\leqslant pK_1 \varepsilon^{1/2} + (1-p) K_2 \varepsilon^{1/2} = C_1 \varepsilon^{1/2} \tag{3}$$

2.2. Lower Bound

From (1) it is clear that if a vector spends a positive fraction of time (asymptotically) in some cone σ_K, then the dilation of the vector as it moves under the action of $\{X_n\}$ is large enough to obtain a lower bound of the form $C_2 \varepsilon^{1/2}$. Note that as a vector gets close to the X axis, the dilations produced by both the A and B matrixes become smaller. Thus, the idea is to show that for a.e. w, the orbit of a vector stays away from the X axis.

Consider the random variables $\{Z_k\}$ defined as follows: $Z_0 = V_0$, where V_0 is some vector in S^1 (the unit circle),

$$Z_k = \frac{X_k Z_{k-1}}{|X_k Z_{k-1}|}, \qquad k = 1, 2, 3,...$$

Clearly, Z_k is an S^1-valued random variable. Independence of $\{X_k\}$ implies that $\{Z_k\}$ is a Markov sequence. Let ν be a stationary initial measure for $\{Z_k\}$. A theorem of Furstenberg and Kifer[6] gives the following formula for λ:

$$\lambda = \operatorname*{Sup}_{\nu} \int d\nu(z) \, [p \ln |Zz| + (1-p) \ln |Bz|]$$

where the Sup is over the set of stationary initial distributions of $\{Z_k\}$. Let ν be a stationary distribution of $\{Z_k\}$ supported on σ. Then

$$\lambda \geqslant \int_{\sigma} d\nu(z) \, [p \ln /Az| + (1-p) \ln |Bz|] \tag{4}$$

From (1) it is clear that if we show that for some $k \in IN$, $\lim_{\varepsilon \to 0} \nu(\sigma_k) > 0$ (both ν and σ_k depend on ε), we can obtain the desired lower bound.

We coordinatize $S^1 \cap \sigma$ by using the slope $m(V)$ of a vector V as its coordinate,

$$m(AV) = \frac{m(V)}{1 + dm(V)}, \qquad m(BV) = \frac{c + (1+a) m(V)}{(1+a) + bm(V)}$$

$$\Delta_A = m(AV) - m(V) = -m_E^2 \frac{d(m/m_E)^2}{1 + m_E d(m/m_E)}$$

$$\Delta_B = m(BV) - m(V) = m_E^2 \frac{1 - (m/m_E)^2}{(1+a)/b + m_E(m/m_E)}$$

where $m = m(v)$.

Thus we obtain a Markov chain $\{Q_n\}$ on $[0, m_E]$ where $\Delta_A(m)$ and $\Delta_B(m)$ are the step sizes to the left and right starting from m. Define a Markov chain $\{Y_n\}$ on $[0, 1]$ by scaling $\{Q_n\}$ as follows:

$$Y_n = Q_n/m_E$$

Recall that $m_E = (c/b)^{1/2} = M(\varepsilon)\varepsilon^{1/2}$, where $\lim_{\varepsilon \to 0} M(\varepsilon) = (c_0/b_0)^{1/2} > 0$. Let

$$L(\varepsilon, X) = \frac{dM(\varepsilon)}{1 + dM(\varepsilon) X\varepsilon^{1/2}}, \qquad R(\varepsilon, X) = \frac{M(\varepsilon)}{[1 + a(\varepsilon)]/b(\varepsilon) + M(\varepsilon) X\varepsilon^{1/2}}$$

The transition probability for $\{Y_n\}$, $P(X, \cdot)$, can be written as

$$P(X, \cdot) = p\delta_{X,(X - LX^2\varepsilon^{1/2})} + (1 - p)\,\delta_{X,(X + R(1 - X^2)\varepsilon^{1/2})}$$

Let π be a stationary initial distribution for $\{Y_n\}$. If f is a bounded, measurable function on $[0, 1]$, then $\int d\pi(x)(Pf - f)(x) = 0$, where

$$Pf(x) = \int P(x, dy)\, f(y)$$

Let $f(x) = x$; then

$$\int d\pi(x)\, [Pf(x) - f(x)] = \varepsilon^{1/2} \int d\pi(x)\, \{(1 - p)R - [(1 - p)R + pL]x^2\}$$

Let

$$g(x) = \varepsilon^{1/2}\{(1 - p)\, R(\varepsilon, x) - [(1 - p)\, R(\varepsilon, x) + pL(\varepsilon, x)]\, x^2\}$$

A simple computation shows that if $M(\varepsilon)\varepsilon < 4$, then $g'(x) < 0$, for all $x \in [0, 1]$. Moreover,

$$g(0) > 0 \qquad \text{and} \qquad g(1) < 0$$

Let $\bar{x}(\varepsilon)$ be the point in $[0, 1]$ where $g(x, \varepsilon)$ crosses the X axis, $\lim_{\varepsilon \to 0} \bar{x}(\varepsilon) = x_0 = \{(1 - p) M_0 b_0/[(1 - p) M_0 b_0 + pd_0 M_0]\}^{1/2}$

$$g\left(\frac{\bar{x}}{2}\right) \pi\left(\left[0, \frac{\bar{x}}{2}\right]\right) \leqslant -\int_{\bar{x}/2}^{1} g(x)\, d\pi \leqslant |g(1)|\, \pi\left(\left[\frac{\bar{x}}{2}, 1\right]\right)$$

Therefore

$$\pi\left(\left[\frac{\bar{x}}{2}, 1\right]\right) \geqslant \frac{g(\bar{x}/2)}{|g(1)| + g(\bar{x}/2)} > 0 \qquad (5)$$

From the definition of Y_n, it is clear that there exists a stationary initial measure ν for $\{Z_n\}$ such that

$$\nu\left(\left\{V\in\sigma\,\middle|\,\frac{\bar{x}}{2}m_E\leqslant m(V)\leqslant m_E\right\}\right)=\pi\left(\left[\frac{\bar{x}}{2},1\right]\right)>0$$

Since $\lim_{\varepsilon\to 0}\bar{x}=x_0>0$, $\lim_{\varepsilon\to 0}(\bar{x}/2)>1/K$ for some $K\in IN$. Thus, we have proved

$$\lim_{\varepsilon\to 0}\nu(\sigma_K)>\frac{g_0(x_0/2)}{pdM_0+g_0(x_0/2)}>0 \tag{6}$$

Theorem. There exists a constant $C_2>0$ such that $\lambda\geqslant C_2\varepsilon^{1/2}$ for small enough ε.

Proof. From (4) we have

$$\lambda\geqslant\int_\sigma d\nu(z)\,[p\ln|Az|+(1-p)\ln|Bz|]$$

$$\geqslant\int_{\sigma_K} d\nu(z)\,[p\ln|Az|+(1-p)\ln|Bz|]$$

where σ_K is as defined in (6). From (1) we have

$$>\int_{\sigma_K} d\nu(z)[p\ln(1+l_1\varepsilon^{1/2})+(1-p)\ln(1+l_2\varepsilon^{1/2})]>C_2\varepsilon^{1/2}\text{ for small enough }\varepsilon$$

We have thus proved

$$\lim_{\varepsilon\to 0}\frac{\ln\lambda(\varepsilon)}{\ln\varepsilon}=\frac{1}{2}$$

3. CONCLUDING REMARKS

There are two other interesting hyperbolic perturbations of a parabolic matrix:

1. If one assumes that $b(\varepsilon)\to 0$ and $c(\varepsilon)\to c_0>0$, then the B matrix limits to a lower triangular matrix. In this case $\lim_{\varepsilon\to 0}\lambda>0$.

2. If one assumes $b(\varepsilon)=b_0\varepsilon+o(\varepsilon)$, $c(\varepsilon)=c_0\varepsilon+o(\varepsilon)$, and $b_0|c_0\to\gamma>0$, then by methods used in this paper one can show that λ scales like ε as $\varepsilon\to 0$.

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